Nonlinear Dyn (2017) 90:2711–2727
DOI 10.1007/s11071-017-3832-3

ORIGINAL PAPER

An asymptotic method for quasi-integrable Hamiltonian system with multi-time-delayed feedback controls under combined Gaussian and Poisson white noises

Wantao Jia · Yong Xu · Zhonghua Liu · Weiqiu Zhu

Received: 4 February 2017 / Accepted: 23 September 2017 / Published online: 6 October 2017

Abstract In the present paper, we consider an approximate approach for predicting the responses of the quasi-integrable Hamiltonian system with multi-time-delayed feedback control under combined Gaussian and Poisson white noise excitations. Two-step approximation is taken here to obtain the responses of such system. First, based on the property of the system solution, the time-delayed system state variables are approximated by using the system state variables without time delay. After this approximation, the system is converted to the one without time delay but with delay time as parameters. Then, stochastic averaging method for quasi-integrable Hamiltonian system under combined Gaussian and Poisson white noises is applied to simplify the converted system to obtain the averaged stochastic integro-differential equations and generalized Fokker–Planck–Kolmogorov equations for both non-resonant and resonant cases. Finally, two examples are worked out to show the detailed procedure of proposed method for the illustrative purpose. And the influences of the time delay on the responses of the systems are also discussed. In addition, the validity of the results obtained by present method is verified by Monte Carlo simulation.

Keywords Quasi-integrable Hamiltonian system · Multi-time-delayed feedback control · Combined Gaussian and Poisson white noise excitations · Stochastic averaging method

W. Jia (✉) · Y. Xu (✉)
Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China
e-mail: jiawantao@nwpu.edu.cn

Y. Xu
e-mail: hsux3@nwpu.edu.cn

Z.Liu
Department of Civil Engineering, Xiamen University, Xiamen 361005, China

W. Zhu
Department of Mechanics, State Key Laboratory of Fluid Power Transmission and Control, Zhejiang University, Hangzhou 310027, China

1 Introduction

In the last few decades, the dynamic and control problems for time-delayed systems have attracted many researchers' attention, and many results have been obtained [1,2]. It is shown that the time delay in the system usually leads to poor performance and complicated dynamics in control systems and even causes the occurrence of some dynamical phenomena, such as bifurcation and chaos [3].

Among the time-delayed systems, the systems with time delay in feedback control force are an important category. In practice, this time delay may be caused by measuring and estimating the system state, calculating and executing the control forces, etc. In past few decades, the system with time delay subjected to deterministic excitations have been studied extensively. Many results can be found in Refs. [4–10]. However, since the existence of disturbance in the environment, the stochastic excitation should be accounted in the system. To improve the deterministic model, the stochastic excitations are added to the time-delayed feedback control systems. So far, for the case of controlled system under stochastic excitation, the results mainly focus on the dynamics and optimal control problem of controlled system under Gaussian white noise excitation [11–15]. In particular, in last decade, Zhu et al. have generalized the stochastic averaging method to study of the systems with time-delayed feedback control under stochastic excitation. The stochastic response, stochastic stability, stochastic reliability, and optimal control problem for nonlinear system with different type of stochastic excitation have been studied by using stochastic averaging method [16–20].

In the previous studies, the stochastic excitation is usually assumed to be stochastic continuous excitation, such as Gaussian white noise. But, in real word, the stochastic excitations are not always continuous. In society, engineering problem, we usually face continuous and discrete stochastic excitation. This type of stochastic excitation is usually modeled as the jump-diffusion process or combined Gaussian and Poisson white noise excitation. Until now, there are a number of researchers working in this area. The stochastic calculus, stochastic differential equations, and the stochastic optimal control problem have been stated systematically in mathematics by Hanson [21] and Øksendal [22]. Zhu et al. have studied the stochastic response and the stochastic stability of the quasi-Hamiltonian system under combined Gaussian white noise and Poisson white noise excitation in terms of stochastic averaging method [23–26]. However, to the authors' knowledge, the stochastic dynamics of system with time-delayed feedback control have not been studied. In particular, the prediction of response such system is still a problem.

In the present paper, we study a tool for predicting the response of quasi-integrable Hamiltonian system with multi-time-delayed feedback control under combined Gaussian and Poisson white noise excitation. We use two-step approximation to get the approximation stationary solution. First, by approximating the system state variables in terms of those without time delay, the feedback control force can be expressed by using the state variables without time delay. At this time, the system with time-delayed variable is transformed to the one without time-delayed system variables but with time delay as parameters. Then, the stochastic averaging method for quasi-Hamiltonian system under combined Gaussian and Poisson white noise excitation can be applied to the converted system. The averaged generalized Fokker–Planck–Kolmogorov (GFPK) equations for both cases, the resonant and non-resonant case, are obtained separately. After solving the averaged GFPK equation, the probability density functions (PDFs) for approximate stationary solution are derived. At last, two examples are calculated for illustrating the application of the proposed method. The Monte Carlo simulation is carried out to show the effectiveness of proposed method.

2 Quasi-integrable Hamiltonian systems with time-delayed feedback control forces

Consider an n-degree-of-freedom (n-DOF) quasi-Hamiltonian system with multi-time-delayed feedback controls under combined Gaussian and Poisson white noise excitations. The equations of motion of system have following form:

$$
\begin{aligned}
\dot{Q}_i &= \frac{\partial H'}{\partial P_i}, \\
\dot{P}_i &= -\frac{\partial H'}{\partial Q_i} - \varepsilon^2 \sum_{j=1}^{n} c_{ij}(\mathbf{Q},\mathbf{P}) \frac{\partial H'}{\partial P_j} \\
&\quad - \varepsilon^2 F_i\left(\mathbf{Q}_{\tau_1}, \mathbf{P}_{\tau_1}, \ldots, \mathbf{Q}_{\tau_s}, \mathbf{P}_{\tau_s}\right) \\
&\quad + \varepsilon \sum_{k=1}^{n_g} g_{ik}(\mathbf{Q},\mathbf{P}) \zeta_k(t) \\
&\quad + \varepsilon \sum_{l=1}^{n_p} h_{il}(\mathbf{Q},\mathbf{P}) \xi_l(t),
\end{aligned}
\tag{1}
$$

where $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_n]^T$ is a vector of generalized displacements and $\mathbf{P} = [P_1, P_2, \ldots, P_n]^T$ is a vector of generalized momenta; $c_{ij}(\mathbf{Q},\mathbf{P})$ denote differentiable functions representing coefficients of quasi linear dampings; $\varepsilon^2 F_i\left(\mathbf{Q}_{\tau_1}, \mathbf{P}_{\tau_1}, \ldots, \mathbf{Q}_{\tau_s}, \mathbf{P}_{\tau_s}\right)$ with $\mathbf{Q}_{\tau_r} = \mathbf{Q}(t-\tau_r)$ and $\mathbf{P}_{\tau_r} = \mathbf{P}(t-\tau_r)$ represent multi-time-delayed feedback controls;$g_{ik}(\mathbf{Q},\mathbf{P})$ are twice differentiable functions representing amplitudes of Gaussian random excitations; $h_{il}(\mathbf{Q},\mathbf{P})$ are infinitely differentiable functions representing amplitudes of Poisson random excitations; ε is a small positive parameter; $\zeta_k(t)$ are Gaussian white noises with zero mean and correlations functions $E[\zeta_k(t)\zeta_l(t +$

$\tau)] = 2D_{kl}\delta(\tau)(k, l = 1, \ldots, n_g)$; $\xi_l(t)$ are independent Poisson white noises [27] with zero mean and the formal derivatives of compound Poisson processes $C_l(t) = \sum_{k=1}^{N_l(t)} Y_{lk} U(t - t_k), l = 1, \ldots, n_p$, where $N_l(t)$ are homogeneous Poisson counting processes with mean arrival rate $\lambda_l > 0$; $U(\cdot)$ is the unit step function; $\{Y_{lk}\}$ are independent identically distributed random variables representing the impulse amplitudes, which are independent of the pulse occurring time t_k.

It is known that the Hamiltonian associated with Eq. (1) (for the case $\varepsilon = 0$) can be non-integrable, integrable, and partially integrable [28]. In the present paper, we assume the associated Hamiltonian system of system (1) is integrable. It means that, there exit n independent integrals of motion $H_1, H_2, \ldots, H_n$ which are in involution in the associated Hamiltonian system.

For the integrable Hamiltonian system, the action-angle variables I_i and θ_i can be introduced by using the following canonical transformations [29]:

$$I_i = I_i(\mathbf{q}, \mathbf{p}), \quad \theta_i = \theta_i(\mathbf{q}, \mathbf{p}), i = 1, 2, \ldots, n$$

Then, the Hamiltonian system can be rewritten as following canonical form:

$$\dot{I}_i = -\frac{\partial}{\partial \theta_i} H(\mathbf{I}) = 0, \quad \dot{\theta}_i = \frac{\partial}{\partial I_i} H(\mathbf{I}) = \omega_i(\mathbf{I}),$$

where $\omega_i(\mathbf{I})$ are the frequencies of the system. An integrable Hamiltonian system is called resonant or non-resonant depending upon the number of the strong resonant relations of form

$$k_i^u \omega_i = 0, \quad u = 1, 2, \ldots, \alpha; \; i = 1, 2, \ldots, n,$$

among the frequencies $\omega_i(\mathbf{I})$, where k_i^u are integers and not all zero for a fixed u and α is the number of resonant relationships. If there is no resonant relation, the system is called non-resonant. If there are $n-1$ resonant relations, namely $\alpha = n - 1$, the system is called completely resonant. If the number of resonant relation is between 1 and $n-1$, namely $1 \le \alpha < n-1$, the system is called partially resonant.

This system can be modeled as the as Stratonovich stochastic differential equations (SDEs) and then transformed into Itô SDEs by adding the correction terms for both Gaussian and Poisson white noise excitation [23,24]:

$$\mathrm{d}Q_i = \frac{\partial H'}{\partial P_i}\mathrm{d}t,$$

$$\begin{aligned}\mathrm{d}P_i = \Bigg[& -\frac{\partial H'}{\partial Q_i} - \varepsilon^2 \sum_{j=1}^{n} c_{ij}(\mathbf{Q}, \mathbf{P}) \frac{\partial H'}{\partial P_j} \\ & + \varepsilon^2 \sum_{k,l=1}^{n_g} \sum_{j=1}^{n} D_{kl} g_{jl}(\mathbf{Q}, \mathbf{P}) \frac{\partial g_{ik}}{\partial P_j} \\ & - \varepsilon^2 F_i\left(\mathbf{Q}_{\tau_1}, \mathbf{P}_{\tau_1}, \ldots, \mathbf{Q}_{\tau_s}, \mathbf{P}_{\tau_s}\right) \Bigg] \mathrm{d}t \\ & + \varepsilon \sum_{k=1}^{n_g} \sigma_{ik}(\mathbf{Q}, \mathbf{P}) \mathrm{d}B_k(t) \\ & + \sum_{l=1}^{n_p} \left(\sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} h_{il}^{(j)}(\mathbf{Q}, \mathbf{P}) (\mathrm{d}C_l(t))^j \right), \end{aligned} \tag{2}$$

where $h_{il}^{(j)}(\mathbf{Q}, \mathbf{P}) = \sum_{s=1}^{n} \frac{\partial h_{il}^{(j-1)}(\mathbf{Q}, \mathbf{P})}{\partial P_s} h_{sl}$, $h_{il}^{(1)}(\mathbf{Q}, \mathbf{P}) = h_{il}(\mathbf{Q}, \mathbf{P})$, $\sigma_{ik} = (\mathbf{gL})_{ik}$, $\mathbf{g} = [g_{jl}]_{n \times n_g}$, $\mathbf{LL}^T = 2\mathbf{D}$. The $\sum_{l=1}^{n_p} \sum_{j=2}^{\infty} \frac{\varepsilon^j}{j!} h_{il}^{(j)}(\mathbf{Q}, \mathbf{P}) (\mathrm{d}C_l(t))^j$ are the correction terms for Poisson white noise excitations in transforming from Stratonovich SDEs into Itô SDEs proposed by Di Paola [30]. The $\varepsilon^2 \sum_{k,l=1}^{n_g} \sum_{j=1}^{n} D_{kl} g_{jl} \frac{\partial g_{ik}}{\partial P_j}$ are the Wong–Zakai correction terms for Gaussian white noise excitations in transforming from Stratonovich SDEs into Itô SDEs [31].

In the following parts, we consider two-step approximation approach to derive the solution of the system.

3 First step approximate

Further, when the Hamiltonian associated with system (1) is assumed to separable and has the following form [28]:

$$H' = \sum_{i=1}^{n} H_i'(q_i, p_i), \quad H_i' = \frac{1}{2} p_i^2 + G(q_i)$$

The time-delayed system state variables can be approximated by [16]:

$$\begin{aligned} Q_i(t - \tau_r) &= Q_i(t) \cos(\omega_i \tau_r) - \frac{P_i(t)}{\omega_i} \sin(\omega_i \tau_r) \\ P_i(t - \tau_r) &= P_i(t) \cos(\omega_i \tau_r) + Q_i(t) \sin(\omega_i \tau_r) \end{aligned} \tag{3}$$

Substituting Eq. (3) to Eq. (2), the force $F_i\left(\mathbf{Q}_{\tau_1}, \mathbf{P}_{\tau_1}, \ldots, \mathbf{Q}_{\tau_s}, \mathbf{P}_{\tau_s}\right)$ can be written as $F_i(\mathbf{Q}, \mathbf{P}; \boldsymbol{\tau})$ ($\boldsymbol{\tau} = [\tau_1, \ldots, \tau_s]$). Then, $\varepsilon^2 \sum_{k,l=1}^{n_g} \sum_{j=1}^{n} D_{kl} g_{jl}(\mathbf{Q}, \mathbf{P}) \frac{\partial g_{ik}}{\partial P_j} - \varepsilon^2 F_i$

$(\mathbf{Q}, \mathbf{P}; \tau)$ can be split into conservative part and dissipated part [16]. These two parts can be combined with $-\frac{\partial H'}{\partial P_i}$ and $-\varepsilon^2 \sum_{j=1}^{n} c_{ij}(\mathbf{Q}, \mathbf{P}) \frac{\partial H'}{\partial P_j}$, respectively. Equation(2) becomes following SIDE

$$\begin{aligned} \mathrm{d}Q_i &= \frac{\partial H}{\partial P_i} \mathrm{d}t, \\ \mathrm{d}P_i &= -\left[\frac{\partial H}{\partial Q_i} + \varepsilon^2 \sum_{j=1}^{n} m_{ij}(\mathbf{Q}, \mathbf{P}; \boldsymbol{\tau}) \frac{\partial H}{\partial P_j}\right] \mathrm{d}t \\ &\quad + \varepsilon \sum_{k=1}^{n_g} \sigma_{ik}(\mathbf{Q}, \mathbf{P}) \mathrm{d}B_k(t) \\ &\quad + \sum_{l=1}^{n_p} \int_{\mathcal{Q}_l} \gamma_{il}(\mathbf{Q}, \mathbf{P}, Y_l) \mathcal{P}_l(\mathrm{d}t, \mathrm{d}Y_l), \end{aligned} \tag{4}$$

$i = 1, 2, \ldots, n;$

where $\gamma_{il}(\mathbf{Q}, \mathbf{P}, Y_l) = \sum_{j=1}^{\infty} \frac{\varepsilon^j}{j!} h_{il}^{(j)}(\mathbf{Q}, \mathbf{P}) Y_l^j$, $H = H(Q, P, \tau)$ and $\mathscr{P}_l(\mathrm{d}t, \mathrm{d}Y_l)$ are Poisson random measures and $\mathcal{Q}_l$ denote the Poisson mark spaces. Suppose that system (4) is still integrable. And this system is a quasi-integrable Hamiltonian system with parameter $\boldsymbol{\tau}$ but without time-delayed control force. Now, the stochastic averaging method for quasi-integrable Hamiltonian systems under combined Poisson and Gaussian white noise can be applied to system (4).

4 Second-step approximation: the stochastic averaging procedure

The stochastic averaging method for quasi-integrable Hamiltonian systems under combined Gaussian and Poisson white noise has been developed by the present authors [23,24]. The dimension and form of averaged equations depend upon the integrability and resonance of associated Hamiltonian system. Thus, in the following parts, the problem will be discussed in non-resonant case and resonant case.

4.1 Non-resonant case

In this case, after introducing the canonical transformations of action-angle variables:

$$I_r = I_r(\mathbf{Q}, \mathbf{P}), \quad \Theta_r = \Theta_r(\mathbf{Q}, \mathbf{P}), r = 1, 2, \ldots, n, \tag{5}$$

Then, the averaged GFPK equation is of the form [23]

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\sum_{r_1=1}^{n} \frac{\partial}{\partial I_{r_1}} \left(\bar{A}_{r_1}(\mathbf{I}) p\right) + \frac{1}{2!} \sum_{r_1, r_2}^{n} \frac{\partial^2}{\partial I_{r_1} \partial I_{r_2}} \left(\bar{A}_{r_1, r_2}(\mathbf{I}) p\right) \\ &\quad - \frac{1}{3!} \sum_{r_1, r_2, r_3=1}^{n} \frac{\partial^3}{\partial I_{r_1} \partial I_{r_2} \partial I_{r_3}} \left(\bar{A}_{r_1, r_2, r_3}(\mathbf{I}) p\right) \\ &\quad + \cdots + (-1)^u \frac{1}{u!} \sum_{r_1, r_2, \ldots, r_u=1}^{n} \frac{\partial^u}{\partial I_{r_1} \partial I_{r_2} \cdots \partial I_{r_u}} \\ &\quad \left(\bar{A}_{r_1, r_2, \ldots, r_u}(\mathbf{I}) p\right) + \mathrm{O}\left(\varepsilon^{u+1}\right), \end{aligned} \tag{6}$$

in which $\bar{A}_{r_i}(\mathbf{I})$, $\bar{A}_{r_i, r_j}(\mathbf{I})$, $\bar{A}_{r_i, r_j, r_k}(\mathbf{I}) \cdots$ are the coefficients of the averaged GFPK equation which have been given in "Appendix A".

In Eq. (6), $p = p(\mathbf{I}, t | \mathbf{I}_0)$, the transition probability density of $\mathbf{I} = [I_1, I_2, \ldots, I_n]^T$ with initial condition

$$p(\mathbf{I}, 0 | \mathbf{I}_0) = \delta(\mathbf{I} - \mathbf{I}_0), \tag{7}$$

or, $p = p(\mathbf{I}, t)$, the probability density of $\mathbf{I}$ with initial condition

$$p(\mathbf{I}, 0) = p(\mathbf{I}_0), \tag{8}$$

depending on whether an initial state or an initial probability density is specified. For stationary situation, the GFPK equation (6) is usually subjected to following boundary conditions:

$$p(\mathbf{I})|_{I_r=0} = \text{finite}, \lim_{I_r \to \infty} p(\mathbf{I}) = 0, \lim_{I_r \to \infty} \frac{\partial^k}{\partial I_r^k} p(\mathbf{I}) = 0, \tag{9}$$

$r = 1, 2, \ldots, n; \quad k = 1, 2, \ldots.$

And it is also subjected to the normalization condition

$$\int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} p(\mathbf{I}) \mathrm{d}I_1 \mathrm{d}I_2 \cdots \mathrm{d}I_n = 1. \tag{10}$$

The exact analytical solution of reduced GFPK equation is very hard to obtain. Usually, the perturbation method or finite difference method are used to get the approximate stationary solution $p(\mathbf{I})$ for the system. Then, the corresponding joint probability density of the generalized displacements and momenta can be obtained from $p(\mathbf{I})$ as follow [29]:

$$\begin{aligned} p(\mathbf{q}, \mathbf{p}) &= p(\mathbf{I}, \Theta) \left|\frac{\partial(\mathbf{I}, \Theta)}{\partial(\mathbf{q}, \mathbf{p})}\right| = p(\Theta | \mathbf{I}) \\ &\quad p(\mathbf{I}) \left|\frac{\partial(\mathbf{I}, \Theta)}{\partial(\mathbf{q}, \mathbf{p})}\right| = \frac{1}{(2\pi)^n} p(\mathbf{I}), \end{aligned} \tag{11}$$

where $|\partial(\mathbf{I}, \Theta) / \partial(\mathbf{q}, \mathbf{p})|$ is the absolute value of the Jacobian determinant of the canonical transformations from $\mathbf{q}, \mathbf{p}$ to $\mathbf{I}, \Theta$.

4.2 Resonant case

In this case, suppose that the integrable Hamiltonian system associated with Eq. (4) is weakly resonant with the following α $(1 \leq \alpha \leq n-1)$ resonant relations

$$k_r^v \omega_r = \mathrm{O}\left(\varepsilon^2\right), \quad v = 1, 2, \ldots, \alpha; \; r = 1, 2, \ldots, n. \tag{12}$$

By introducing the following combinations Ψ_v of angle variables

$$\Psi_v = k_r^v \Theta_r, \quad v = 1, 2, \ldots, \alpha; \; r = 1, 2, \ldots, n. \tag{13}$$

The following averaged GFPK equation [24] can be obtained:

$$\begin{aligned}
\frac{\partial}{\partial t} p = &- \sum_{r_1=1}^{n} \frac{\partial}{\partial I_{r_1}} \left(\bar{A}_{r_1} (\mathbf{I}, \boldsymbol{\psi})\, p\right) \\
&- \sum_{v_1=1}^{\alpha} \frac{\partial}{\partial \psi_{v_1}} \left(\bar{A}_{n+v_1} (\mathbf{I}, \boldsymbol{\psi})\, p\right) \\
&+ \frac{1}{2} \sum_{r_1, r_2=1}^{n} \frac{\partial^2}{\partial I_{r_1} \partial I_{r_2}} \left(\bar{A}_{r_1, r_2} (\mathbf{I}, \boldsymbol{\psi})\, p\right) \\
&+ \frac{C_2^1}{2} \sum_{v_1=1}^{\alpha} \sum_{r_1=1}^{n} \frac{\partial^2}{\partial I_{r_1} \partial \psi_{v_1}} \left(\bar{A}_{r_1, n+v_1} (\mathbf{I}, \boldsymbol{\psi})\, p\right) \\
&+ \frac{1}{2} \sum_{v_1, v_2=1}^{\alpha} \frac{\partial^2}{\partial \psi_{v_1} \partial \psi_{v_2}} \left(\bar{A}_{n+v_1, n+v_2} (\mathbf{I}, \boldsymbol{\psi})\, p\right) + \cdots \\
&+ \sum_{j=3}^{u} (-1)^j \sum_{s=0}^{j} \sum_{v_1, v_2, \ldots, v_s=1}^{\alpha} \sum_{r_1, r_2, \ldots, r_{j-s}=1}^{n} \\
&\frac{C_j^s}{j!} \frac{\partial^j}{\partial I_{r_1} \cdots \partial I_{r_{j-s}} \partial \psi_{v_1} \cdots \partial \psi_{v_s}} \\
&\left(\bar{A}_{r_1, \ldots, r_{j-s}, n+v_1, \ldots, n+v_s} (\mathbf{I}, \boldsymbol{\psi})\, p\right) + \mathrm{O}\left(\varepsilon^{u+1}\right)
\end{aligned} \tag{14}$$

where $C_j^s = \frac{j!}{s!(m-s)!}$ and $\bar{A}_{r_1} (\mathbf{I}, \boldsymbol{\psi})$, $\bar{A}_{n+v_1} (\mathbf{I}, \boldsymbol{\psi})$, $\bar{A}_{r_1, r_2} (\mathbf{I}, \boldsymbol{\psi})$, $\bar{A}_{r_1, n+v_1} (\mathbf{I}, \boldsymbol{\psi}) \cdots$ are the coefficients of the GFPK equation. And the detailed forms of the coefficients are given in "Appendix B".

In Eq. (14), $p = p\left(\mathbf{I}, \boldsymbol{\psi}, t \,\middle|\, \mathbf{I}_0, \boldsymbol{\psi}_0\right)$ denotes the transition probability density of $[\mathbf{I}, \boldsymbol{\psi}]^T = [I_1, \ldots, I_n, \psi_1, \ldots, \psi_\alpha]^T$ with following initial conditions:

$$p\left(\mathbf{I}, \boldsymbol{\psi}, 0 \,\middle|\, \mathbf{I}_0, \boldsymbol{\psi}_0\right) = \delta \left(\mathbf{I} - \mathbf{I}_0\right) \delta \left(\boldsymbol{\psi} - \boldsymbol{\psi}_0\right) \tag{15}$$

or $p = p\,(\mathbf{I}, \boldsymbol{\psi}, t)$ denotes the probability density of $[\mathbf{I}, \boldsymbol{\psi}]^T$ with initial condition

$$p\,(\mathbf{I}, \boldsymbol{\psi}, 0) = p\left(\mathbf{I}, \boldsymbol{\psi}_0\right) \tag{16}$$

The averaged GFPK equation (14) is also subjected to certain boundary conditions.

Considering the stationary situation, $p = p\,(\mathbf{I}, \boldsymbol{\psi})$ in Eq. (14) is the stationary joint probability density . The boundary condition with respect to $\mathbf{I}$ is

$$p = \text{finite} \quad p \to 0 \quad \text{and} \quad \frac{\partial^k}{\partial I_r^k} p \to 0 \quad \text{as} \quad I_r \to \infty \tag{17}$$

$r = 1, \ldots, n; \; k = 1, 2, \ldots$

Since $p\,(\mathbf{I}, \boldsymbol{\psi})$ is a periodic function of $\boldsymbol{\psi}$, it satisfies the following periodic boundary condition with respect to ψ_v:

$$p|_{\psi_v + 2n\pi} = p|_{\psi_v} \quad \text{and} \quad \frac{\partial^k}{\partial \psi_v^k} p \,\bigg|_{\psi_v + 2n\pi} = \frac{\partial^k}{\partial \psi_v^k} p \,\bigg|_{\psi_v} \tag{18}$$

In addition, the following normalization condition is also satisfied

$$\int_0^{\infty} \cdots \int_0^{\infty} \int_0^{2\pi} \cdots \int_0^{2\pi} p\,(\mathbf{I}, \boldsymbol{\psi}) \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_\alpha \mathrm{d}I_1 \cdots \mathrm{d}I_n = 1 \tag{19}$$

Equation (17) implies that I_r are reflecting boundary. Usually, the same techniques for non-resonant are applied here to get the approximated solution of the averaged reduced GFPK Eq. (14). Then, the stationary solution of averaged GPFK equation in terms of $\mathbf{q}$ and $\mathbf{p}$ can be given as [29]

$$\begin{aligned}
p\,(\mathbf{q}, \mathbf{p}) &= p\,(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1) \left| \frac{\partial\,(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial\,(\mathbf{q}, \mathbf{p})} \right| \\
&= p\,(\boldsymbol{\theta}_1 | \mathbf{I}, \boldsymbol{\psi})\, p\,(\mathbf{I}, \boldsymbol{\psi}) \left| \frac{\partial\,(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial\,(\mathbf{q}, \mathbf{p})} \right| \\
&= \frac{1}{(2\pi)^{n-\alpha}} p\,(\mathbf{I}, \boldsymbol{\psi}) \left| \frac{\partial\,(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)}{\partial\,(\mathbf{q}, \mathbf{p})} \right|
\end{aligned} \tag{20}$$

where $|\partial\,(\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1)/\partial\,(\mathbf{q}, \mathbf{p})|$ is the absolute value of the Jacobian determinant for the transformations from $\mathbf{q}, \mathbf{p}$ to $\mathbf{I}, \boldsymbol{\psi}, \boldsymbol{\theta}_1$.

5 Two examples

5.1 Example 1

Consider a van der Pol oscillator with two time-delayed feedback control subject to Poisson white noise excitations. The equation of motion is

$$\ddot{X} + \omega'^2 X - \varepsilon^2 \left(1 - X^2\right) \dot{X} = -\varepsilon^2 \left(a_1 X_{\tau_1} + a_2 \dot{X}_{\tau_2}\right) + \varepsilon \xi\,(t) \tag{21}$$

where $X_{\tau_1} = X(t - \tau_1)$ and $\dot{X}_{\tau_2} = \dot{X}(t - \tau_2)$ are system variables with time delay; $\varepsilon^2 a_1$ and $\varepsilon^2 a_2$ are feedback control gains; τ_1 and τ_2 are two delay times; $\xi(t)$ is a combined Gaussian white noise and Poisson white noise; ε is a small parameter.

Let $X = Q$ and $\dot{X} = P$ ($X_{\tau_1} = Q_{\tau_1}$ and $\dot{X}_{\tau_2} = P_{\tau_2}$). Thus, system (21) can be written as

$$\begin{aligned}\dot{Q} &= P\\ \dot{P} &= -\omega'^2 Q + \varepsilon^2 \left(1 - Q^2\right) P - \varepsilon^2 \left(a_1 Q_{\tau_1} + a_2 P_{\tau_2}\right)\\ &\quad + \varepsilon\xi(t)\end{aligned} \tag{22}$$

As state in the Sect. 3, the time-delayed system state variables can be replaced approximately by using the system state variables without time delay as following forms:

$$\begin{aligned}Q_{\tau_1} &= Q\cos\left(\omega'\tau_1\right) - \frac{1}{\omega'} P\sin\left(\omega'\tau_1\right)\\ P_{\tau_2} &= Q\omega'\sin\left(\omega'\tau_2\right) + P\cos\left(\omega'\tau_2\right)\end{aligned} \tag{23}$$

Substituting Eq. (23) to Eq. (22), the following equations can be derived:

$$\begin{aligned}\dot{Q} &= P\\ \dot{P} &= -\omega^2(\tau_1, \tau_2) Q + \varepsilon^2 \left[c(\tau_1, \tau_2) - Q^2\right]\\ &\quad P + \varepsilon\xi(t)\end{aligned} \tag{24}$$

In which

$$\omega = \omega(\tau_1, \tau_2) = \sqrt{\omega'^2 + \varepsilon^2 \left(a_1\cos\left(\omega'\tau_1\right) + a_2\sin\left(\omega'\tau_2\right)\right)}$$

$$c = c(\tau_1, \tau_2) = 1 - \left(a_2\cos\left(\omega'\tau_2\right) - \frac{a_1}{\omega'}\sin\left(\omega'\tau_1\right)\right)$$

Thus, the modified Hamiltonian system associated with Eq. (24) is

$$H = \omega I = \frac{p^2 + \omega^2 q^2}{2},$$

By using the approximate procedure stated in Sects. 3 and 5 and ignoring the terms higher than ε^4, the following reduced averaged GFPK equation can be obtained:

$$\begin{aligned}0 &= -\frac{\partial}{\partial I}\left(\bar{A}_1(I)\, p\right) + \frac{1}{2}\frac{\partial^2}{\partial I^2}\left(\bar{A}_2(I)\, p\right)\\ &\quad - \frac{1}{3!}\frac{\partial^3}{\partial I^3}\left(\bar{A}_3(I)\, p\right) + \frac{1}{4!}\frac{\partial^4}{\partial I^4}\left(\bar{A}_4(I)\, p\right)\end{aligned} \tag{25}$$

where

$$\begin{aligned}\bar{A}_1(I) &= -\frac{\varepsilon^2}{2\omega} I^2 + \varepsilon^2 c(\tau_1, \tau_2) I + \varepsilon^2 \frac{2D + \lambda E\left[Y^2\right]}{2\omega}\\ \bar{A}_2(I) &= \varepsilon^2 \frac{2D + \lambda E\left[Y^2\right]}{\omega} I + \varepsilon^4 \frac{\lambda E\left[Y^4\right]}{4\omega^2}\\ \bar{A}_3(I) &= \varepsilon^4 \frac{3}{2}\frac{\lambda E\left[Y^4\right]}{\omega^2} I;\ \bar{A}_4(I) = \varepsilon^4 \frac{3}{2}\frac{\lambda E\left[Y^4\right]}{\omega^4} I^2\end{aligned} \tag{26}$$

where $p = p(I)$.

This reduced averaging GFPK equation (25) can be solved by perturbation method. First, the solution is assumed as following form:

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \tag{27}$$

in which $p_0 = p_0(I)$, $p_1 = p_1(I)$, $p_2 = p_2(I)$. The details of perturbation procedure can be found in "Appendix C".

Further, the stationary probability density of displacement and velocity of original system (21) is

$$p(q, p) = \frac{1}{2\pi}\, p(I)\Big|_{I = \frac{\left(p^2 + \omega^2 q^2\right)}{2\omega}} \tag{28}$$

where $q = x$, $p = \dot{x}$. The marginal stationary probability density and moments can be derived from $p(q, p)$.

To see accuracy of the theoretical method, the stationary probability density for the response is calculated with system parameters: $\varepsilon = 0.1$, $\omega' = 1.2$, $\alpha = 1.0$, $\beta = 1.0$, $a_1 = -3.0$, $a_2 = 3.0$, $\tau_1 = 1.0$, $\tau_2 = 3.0$, $\sigma^2 = 1.0$, $\lambda = 1.0$, $E[Y^2] = 1.0$. In Fig. 1a, the stationary response of displacement is given, and Fig. 1b shows the stationary probability density of the velocity. In these figures, the solid lines denote the theoretical results and the discrete lines denote the Monte Carlo simulation. It can be seen that the analytical results agree well with those from Monte Carlo simulation.

Figures 2 and 3 show the influences of displacement feedback controls and the velocity feedback control on the stationary probability density, respectively. For these two cases, the value of parameters a_1 or a_2 are assumed to be 0, successfully. In these figures, the results obtained by using the proposed stochastic averaging method agree well with those with Monte Carlo simulation. It is seen in Fig. 2a, b that when the value τ_1 changes from 1 to 5, the number of peaks of PDF changes from 1 to 2 (see Fig. 1). If we continue to increase the value of τ_1 from 5 to 7, the number of peaks of PDF changes from 2 to 1. The same phenomena can be observed from Fig. 3a, b. This implies that time-delayed feedback control force may cause the phenomenological bifurcation.

5.2 Example 2

Consider two nonlinear damping oscillators coupled by linear dampings and subject to both external excitation of combined Gaussian and Poisson white noises

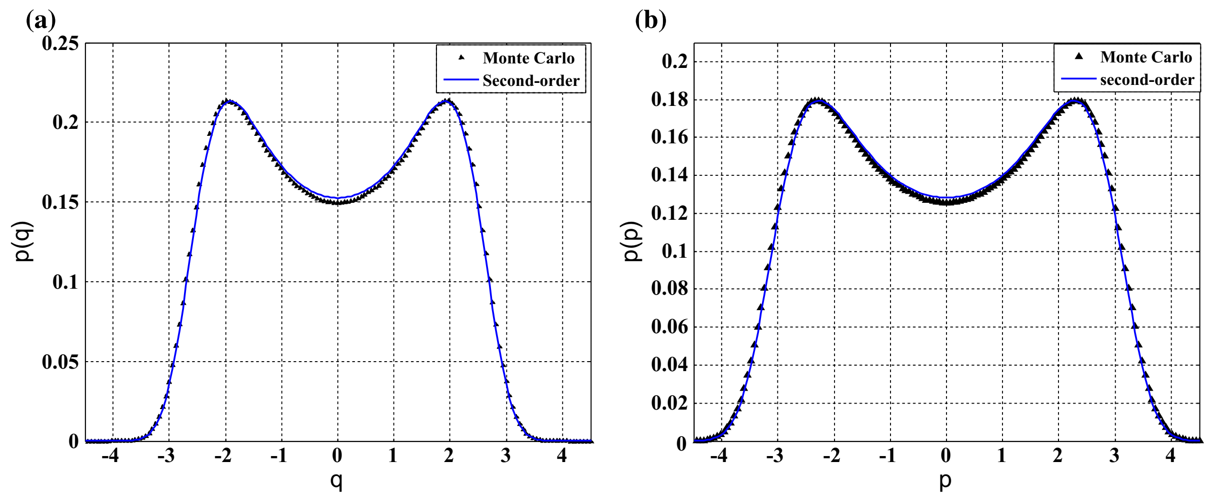

Fig. 1 The marginal probability density function for the displacement and velocity

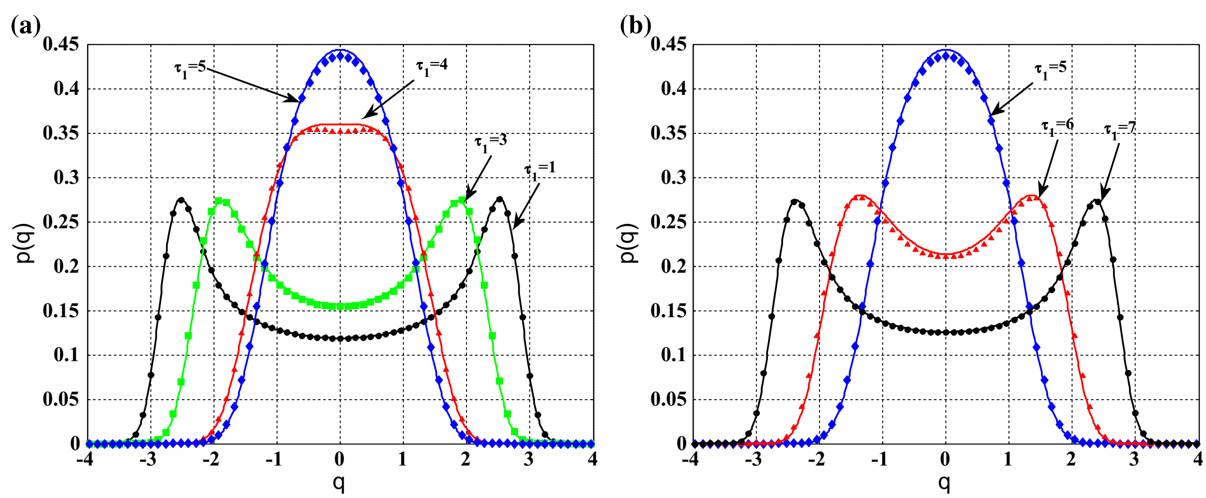

Fig. 2 Stationary marginal probability density of system (21) with displacement feedback. The parameters are: $\omega' = 1.0$, $\varepsilon = 0.1$, $a_1 = 1.0$, $a_2 = 0.0$, $\tau_2 = 1.0$, $\lambda = 0.3$, $E[Y^2] = 1.0$, $\sigma^2 = 0.2$

and 2-time-delayed feedback controls. The equations of motion of the system are of the form:

$$\begin{aligned}
&\ddot{X}_1 + \varepsilon^2(\alpha'_{11} + \alpha_{12}\dot{X}_1^2)\dot{X}_1 + \varepsilon^2\beta_1\dot{X}_2 + \omega'^2_1 X_1 \\
&\quad = u_1 + \varepsilon\left(\zeta_1(t) + \xi_1(t)\right), \\
&\ddot{X}_2 + \varepsilon^2(\alpha'_{21} + \alpha_{22}\dot{X}_2^2)\dot{X}_2 + \varepsilon^2\beta_2\dot{X}_1 + \omega'^2_2 X_2 \\
&\quad = u_2 + \varepsilon\left(\zeta_2(t) + \xi_2(t)\right),
\end{aligned} \tag{29}$$

where ω'_i, α'_{ii}, α_{ij}, β_i $(i, j = 1, 2)$ are constants; ε is small parameter; $u_i = -\varepsilon^2\left(\eta_{i1}X_{i\tau_1} + \eta_{i2}\dot{X}_{i\tau_2}\right) = -\varepsilon^2\left(\eta_{i1}X_i\,(t-\tau_1) + \eta_{i2}\dot{X}_i\,(t-\tau_2)\right)$; $\zeta_i(t)$ $(i = 1, 2)$ are two independent Gaussian white noises with small intensities $2D_{ii}$ $(i = 1, 2)$; ξ_i $(i = 1, 2)$ are two independent Poisson white noises with zero mean and with Gaussian distribution of impulse strength $\lambda_i E[Y_i^2]$. Also, Gaussian white noises are independent of Poisson white noises.

Let $Q_i = X_i$, $P_i = \dot{X}_i$. The time-delayed system state variables can be approximated in terms of those without time delay as

$$\begin{aligned}
Q_{i\tau_1} &= Q_i \cos\omega'_i\tau_1 - \frac{P_i}{\omega'_i}\sin\omega'_i\tau_1 \\
P_{i\tau_2} &= P_i \cos\omega'_i\tau_2 + Q_i\omega'_i \sin\omega'_i\tau_2 \quad i = 1, 2
\end{aligned} \tag{30}$$

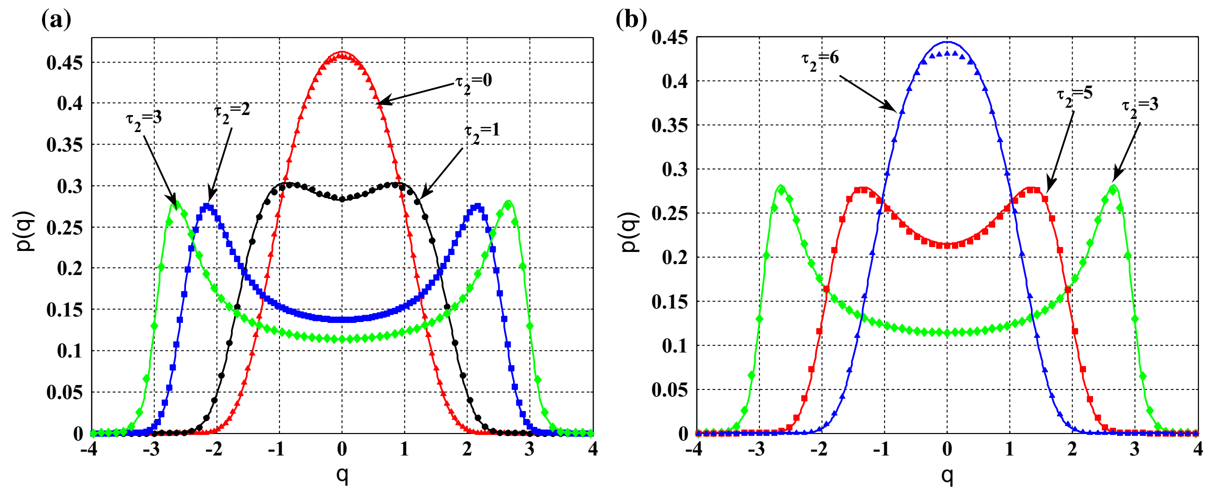

Fig. 3 Stationary marginal probability density of system (21) with velocity feedback. The parameters are: $\omega' = 1.0$, $\varepsilon = 0.1$, $a_1 = 0.0$, $a_2 = 1.0$, $\tau_1 = 1.0$, $\lambda = 0.3$, $E[Y^2] = 1.0$, $\sigma^2 = 0.2$

Thus, the modified system can be expressed as

$$\begin{aligned} \mathrm{d}Q_1 &= P_1 \mathrm{d}t, \\ \mathrm{d}P_1 &= -\left[\omega_1^2 Q_1 + \varepsilon^2\left(\alpha_{11} + \alpha_{12} P_1^2\right) P_1 + \varepsilon^2 \beta_1 P_2\right] \mathrm{d}t \\ &\quad + \varepsilon\left(\sqrt{2D_{11}} \mathrm{d}B_1(t) + \int_{\mathcal{Q}_1} Y_1\, \mathcal{P}_1(\mathrm{d}t, \mathrm{d}Y_1)\right), \\ \mathrm{d}Q_2 &= P_2 \mathrm{d}t, \\ \mathrm{d}P_2 &= -\left[\omega_2^2 Q_2 + \varepsilon^2\left(\alpha_{21} + \alpha_{22} P_2^2\right) P_2 + \varepsilon^2 \beta_2 P_1\right] \mathrm{d}t \\ &\quad + \varepsilon\left(\sqrt{2D_{22}} \mathrm{d}B_2(t) + \int_{\mathcal{Q}_2} Y_2 P_2(\mathrm{d}t, \mathrm{d}Y_2)\right), \end{aligned} \tag{31}$$

where $\omega_i^2 = \omega_i'^2 + \varepsilon^2\left(\eta_{i1}\cos\omega'_i\tau_1 + \eta_{i2}\omega'_i \sin\omega'_i\tau_2\right)$. And the damping coefficients α'_{i1} become $\alpha_{i1} = \alpha'_{i1} + \eta_{i2}\cos\omega'_i\tau_2 - \frac{\eta_{i1}}{\omega'_i}\sin\omega'_i\tau_1$. The Hamiltonian associated with the modified system (31) is:

$$H = \sum_{i=1}^{2} \omega_i I_i, \tag{32}$$

where

$$I_i = \frac{1}{2\omega_i}(p_i^2 + \omega_i^2 q_i^2),\ i = 1, 2. \tag{33}$$

And the corresponding angle variables are

$$\Theta_i = -\arctan\left(\frac{p_i}{\omega_i q_i}\right), \quad i = 1, 2. \tag{34}$$

The system (31) is a quasi-integrable Hamiltonian system. Thus, in the following part, we will discuss the response of the system in non-resonant and resonant case.

In non-resonant case, where $r\omega_1 + s\omega_2 \neq 0$, r, s are integers, neglecting the terms of higher order than fourth order, the averaged GFPK equation is

$$\begin{aligned} \frac{\partial}{\partial t} p &= -\frac{\partial}{\partial I_1}(\bar{A}_1(I_1, I_2)\, p) - \frac{\partial}{\partial I_2}(\bar{A}_2(I_1, I_2)\, p) \\ &\quad + \frac{1}{2!}\frac{\partial^2}{\partial I_1^2}(\bar{A}_{1,1}(I_1, I_2)\, p) \\ &\quad + \frac{1}{2!}\frac{\partial^2}{\partial I_2^2}(\bar{A}_{2,2}(I_1, I_2)\, p) \\ &\quad - \frac{1}{3!}\frac{\partial^3}{\partial I_1^3}(\bar{A}_{1,1,1}(I_1, I_2)\, p) \\ &\quad - \frac{1}{3!}\frac{\partial^3}{\partial I_2^3}(\bar{A}_{2,2,2}(I_1, I_2)\, p) \\ &\quad + \frac{1}{4!}\frac{\partial^4}{\partial I_1^4}(\bar{A}_{1,1,1,1}(I_1, I_2)\, p) \\ &\quad + \frac{1}{4!}\frac{\partial^4}{\partial I_2^4}(\bar{A}_{2,2,2,2}(I_1, I_2)\, p). \end{aligned} \tag{35}$$

where $p = p(I_1, I_2)$ and the coefficients of the averaged GFPK equation are

$$\begin{aligned} \bar{A}_1(I_1, I_2) &= -\varepsilon^2\alpha_{11} I_1 - \frac{3}{2}\varepsilon^2\alpha_{12}\omega_1 I_1^2 \\ &\quad + \frac{\varepsilon^2}{2\omega_1}\left(2D_{11} + \lambda_1 E[Y_1^2]\right), \\ \bar{A}_2(I_1, I_2) &= -\varepsilon^2\alpha_{21} I_2 - \frac{3}{2}\varepsilon^2\alpha_{22}\omega_2 I_2^2 \\ &\quad + \frac{\varepsilon^2}{2\omega_2}\left(2D_{22} + \lambda_2 E[Y_2^2]\right), \end{aligned} \tag{36}$$

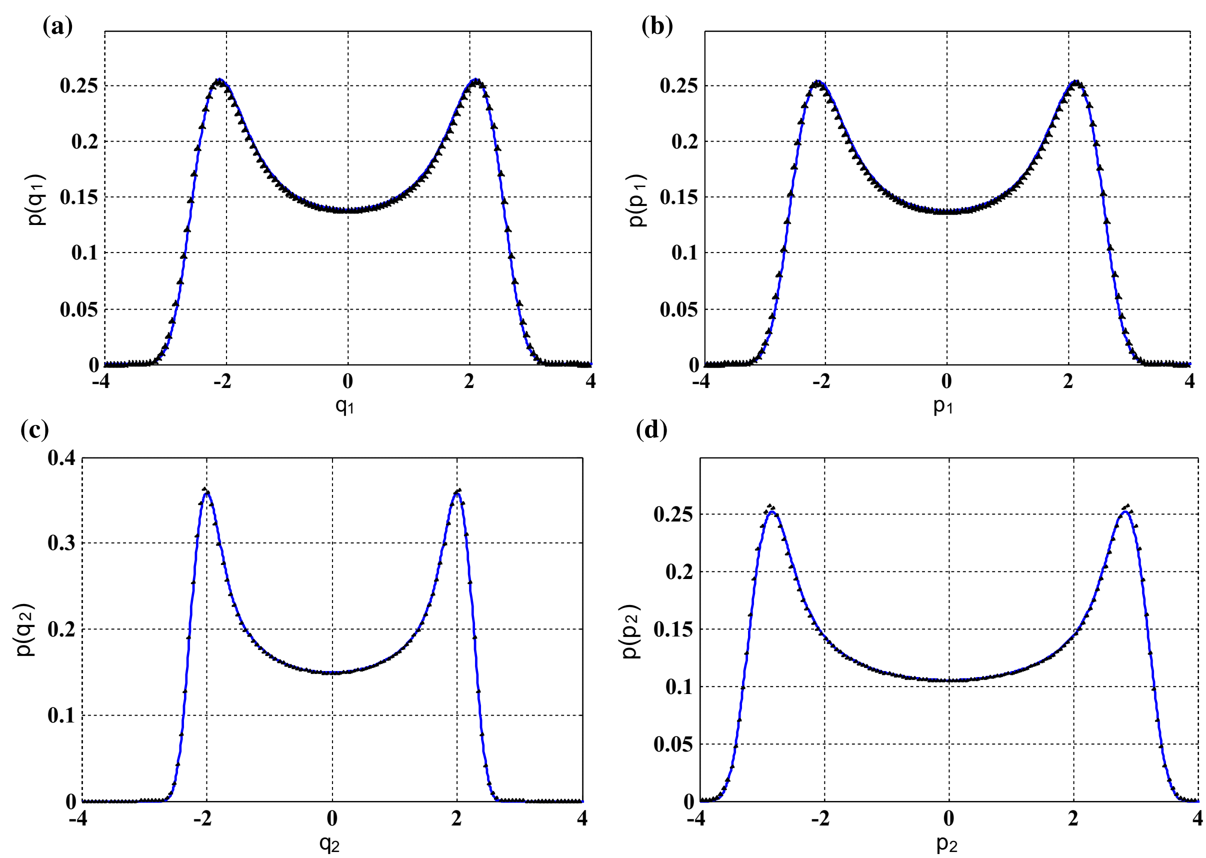

Fig. 4 Stationary marginal probability densities with both displacement and velocity feedback control in non-resonant case

$$\bar{A}_{1,1}(I_1, I_2) = \frac{\varepsilon^2}{\omega_1}\left(2D_{11} + \lambda_1 E[Y_1^2]\right) I_1 + \frac{\varepsilon^4}{4\omega_1^2}\lambda_1 E[Y_1^4],\ \bar{A}_{2,2}(I_1, I_2) = \frac{\varepsilon^2}{\omega_2}\left(2D_{22} + \lambda_2 E[Y_2^2]\right) I_2 + \frac{\varepsilon^4}{4\omega_2^2}\lambda_2 E[Y_2^4];\ \text{otherwise}:\ \bar{A}_{r_1,r_2}(I_1, I_2) = 0, \tag{37}$$

$$\bar{A}_{1,1,1}(I_1, I_2) = \frac{3\varepsilon^4}{2\omega_1^2}\lambda_1 E[Y_1^4] I_1,\ \bar{A}_{2,2,2}(I_1, I_2) = \frac{3\varepsilon^4}{2\omega_2^2}\lambda_2 E[Y_2^4] I_2,\ \text{otherwise}:\ \bar{A}_{r_1,r_2,r_3}(I_1, I_2) = 0, \tag{38}$$

$$\bar{A}_{1,1,1,1}(I_1, I_2) = \frac{3\varepsilon^4}{2\omega_1^2}\lambda_1 E[Y_1^4] I_1^2,\ \bar{A}_{2,2,2,2}(I_1, I_2) = \frac{3\varepsilon^4}{2\omega_2^2}\lambda_2 E[Y_2^4] I_2^2,\ \text{otherwise}:\ \bar{A}_{r_1,r_2,r_3,r_4}(I_1, I_2) = 0. \tag{39}$$

The averaged GFPK equation (35) can be solved by using the perturbation method. After getting the approximate stationary solution $p(I_1, I_2)$, the approximate stationary probability density of the displacement and velocities of original system (29) is then obtained

$$p(q_1, p_1, q_2, p_2) = \frac{1}{4\pi^2}\ p(I_1, I_2)\big|_{I_i=(p_i^2+\omega_i^2 q_i^2)/(2\omega_i^2)}. \tag{40}$$

where $q_1 = x_1, p_1 = \dot{x}_1, q_2 = x_2, p_2 = \dot{x}_2$. The marginal stationary probability density and moments can be obtained from $p(q_1, p_1, q_2, p_2)$.

In order to see the accuracy of proposed method, the responses of system are calculated for the parameter $\varepsilon = 0.1, \omega'_1 = 1.0, \omega'_2 = 1.414, \alpha'_{11} = -6.0, \alpha_{12} =$

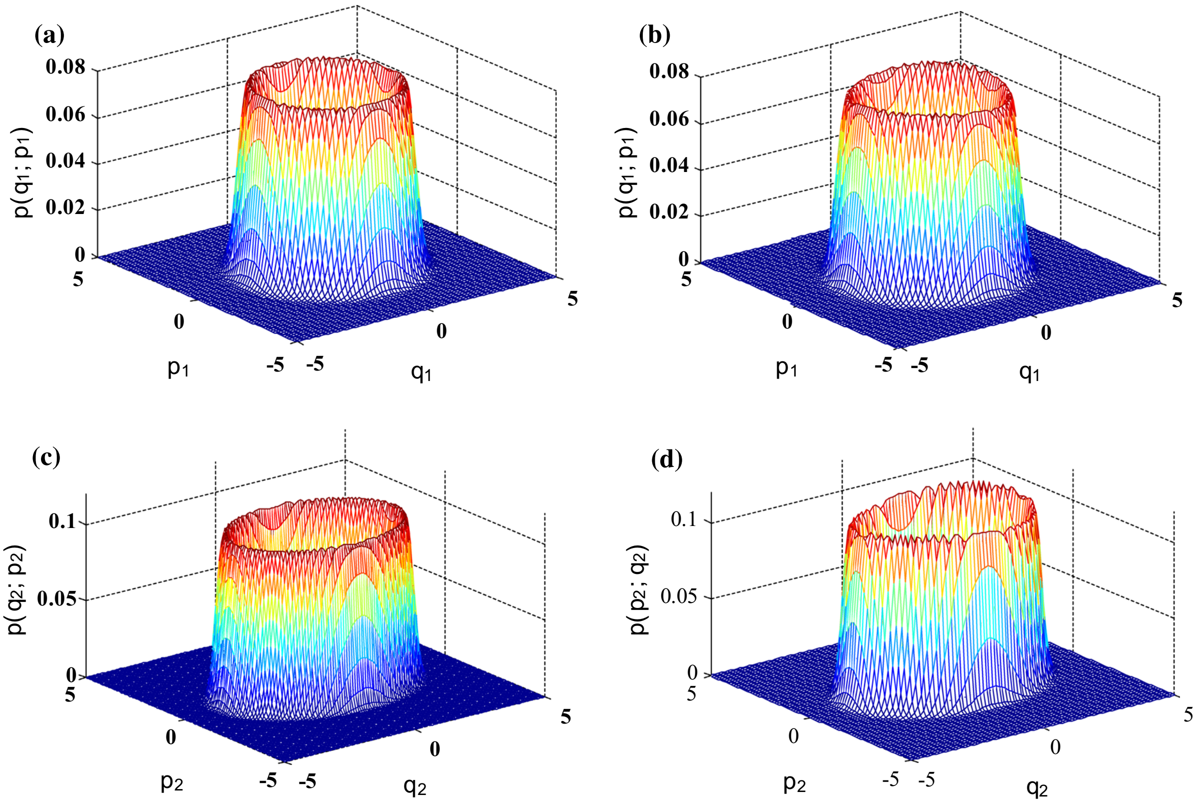

Fig. 5 Stationary joint probability densities $p\,(q_1, p_1)$ and $p\,(q_2, p_2)$ of system with both displacement and velocity feedback control in non-resonant case. **a**, **c** Analytical results and **b**, **d** are results from the Monte Carlo simulation. The parameters are the same as those in Fig. 4.

$1.0, \alpha'_{21} = -6.0, \alpha_{22} = 1.0, \beta_1 = 1.0, \beta_2 = 1.0, \eta_{11} = -4.0, \eta_{12} = 4.0, \eta_{21} = -4.0, \eta_{22} = 4.0, \tau_1 = 1.0, \tau_2 = 2.0, 2D_{11} = 1.0, \lambda_1 = 0.4, E[Y_1^2] = 2.5, 2D_{22} = 1.0, \lambda_2 = 0.4, E[Y_2^2] = 2.5$. The stationary marginal probability density functions for displacements q_i and the velocity p_i are given in Fig. 4. In this figure, the discrete point lines denote the Monte Carlo simulation and the solid lines denote the analytical solutions. Also, the joint probability density functions for displacement and velocity are given in Fig. 5. Shown in Fig. 5a, c are the analytical results and shown in Fig. 5b, d are the results from the Monte Carlo simulation. It can be seen that the analytical results agree well with the Monte Carlo simulation.

In order to see the influence of the τ_1, the probability density function of $p\,(q_1)$ for different values of τ_1 is calculated. The results for the system with time-delayed feedback control forces compared with those without time-delayed feedback control force are given in Fig. 6a–d. In these figures, the blue lines represent results of system with time-delayed feedback control forces, and the red ones denote the results of system without time-delayed control forces. It can be seen that, with the increment of the value of τ_1, the number of peaks of PDFs of stationary response q_1 of system with time-delayed control force changes from 1 to 2, which implies the phenomenological bifurcation occurs. Also, it can be seen that the PDFs of the uncontrolled system remain unchanged, since the vanishing of controlled force term. In these figures, the solid lines denote the theoretical results and the discrete lines denote the Monte Carlo simulation. The two results agree well with each other. The system parameters are given as $\varepsilon = 0.1, \omega'_1 = 1.0, \omega'_2 = 1.414, \alpha'_{11} = -3.0, \alpha_{12} = 8.0, \alpha'_{21} = -3.0, \alpha_{22} = 8.0, \beta_1 = 1.0, \beta_2 = 1.0, \eta_{11} = -3.0, \eta_{12} = 3.0, \eta_{21} = -3.0, \eta_{22} = 3.0, \tau_2 = 4.0, 2D_{11} = 1.0, \lambda_1 = 0.1, E[Y_1^2] = 10.0, 2D_{22} = 1.0, \lambda_2 = 1.0, E[Y_2^2] = 10$.

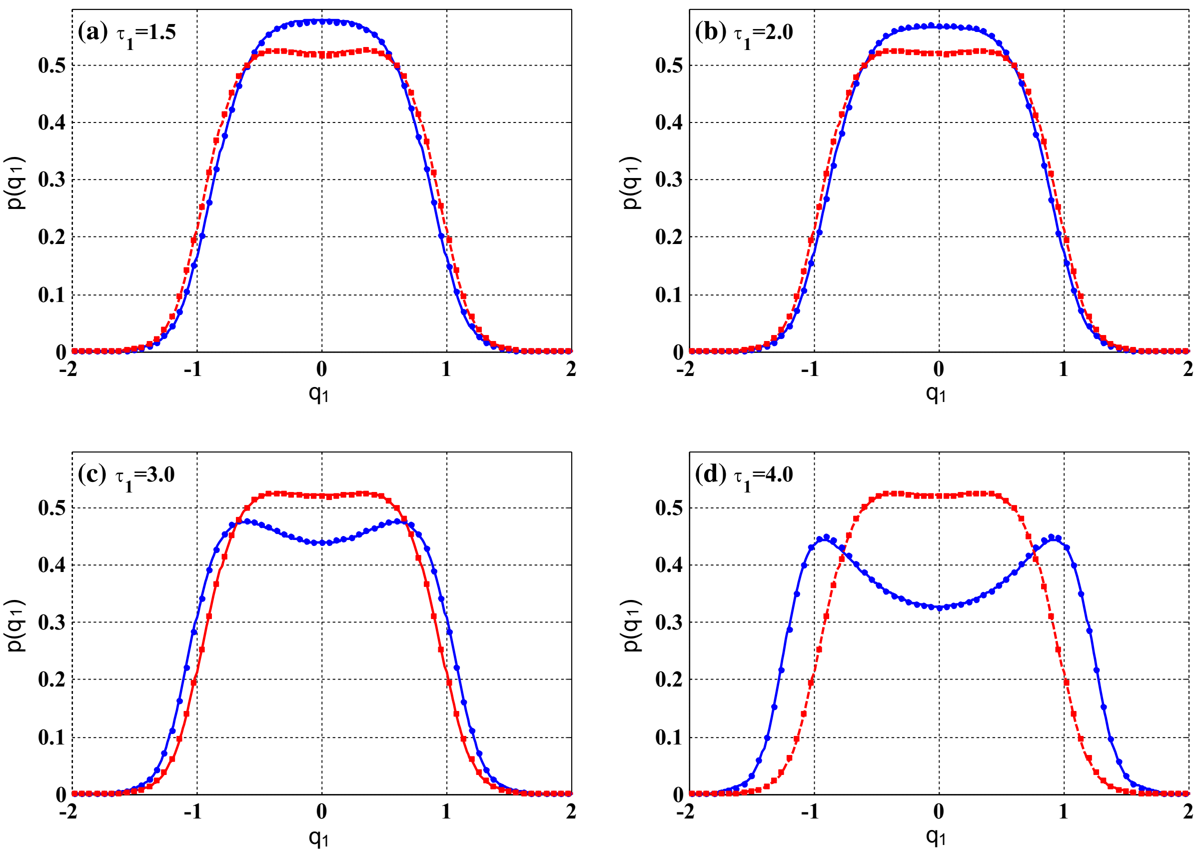

Fig. 6 The influence of τ_1 on the stationary marginal probability density of displacement q_1. The solid lines in this figure are the results from proposed method. The dotted lines in this figure are the results from Monte Carlo simulation

In primary internal resonance, i.e., $\omega_1 = \omega_2 = \omega$. Introduce the combination of angle variables $\psi = \theta_1 - \theta_2$. Thus, the averaged GFPK equation in this case has following form:

$$
\begin{aligned}
0 = & -\frac{\partial}{\partial I_1}\left(\bar{A}_1(I_1, I_2, \psi)\, p\right) - \frac{\partial}{\partial I_2}\left(\bar{A}_2(I_1, I_2, \psi)\, p\right) \\
& -\frac{\partial}{\partial \psi}\left(\bar{A}_3(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{2}\frac{\partial^2}{\partial I_1^2}\left(\bar{A}_{1,1}(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{2}\frac{\partial^2}{\partial I_2^2}\left(\bar{A}_{2,2}(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{2}\frac{\partial^2}{\partial \psi^2}\left(\bar{A}_{3,3}(I_1, I_2, \psi)\, p\right) \\
& -\frac{1}{3!}\frac{\partial^3}{\partial I_1^3}\left(\bar{A}_{1,1,1}(I_1, I_2, \psi)\, p\right) \\
& -\frac{1}{3!}\frac{\partial^3}{\partial I_2^3}\left(\bar{A}_{2,2,2}(I_1, I_2, \psi)\, p\right) \\
& -\frac{1}{3!}\frac{\partial^3}{\partial \psi^3}\left(\bar{A}_{3,3,3}(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{4!}\frac{\partial^4}{\partial I_1^4}\left(\bar{A}_{1,1,1,1}(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{4!}\frac{\partial^4}{\partial I_2^4}\left(\bar{A}_{2,2,2,2}(I_1, I_2, \psi)\, p\right) \\
& +\frac{1}{4!}\frac{\partial^4}{\partial \psi^4}\left(\bar{A}_{3,3,3,3}(I_1, I_2, \psi)\, p\right) \\
& +\frac{6}{4!}\frac{\partial^4}{\partial I_1^2 \partial \psi^2}\left(\bar{A}_{1,1,3,3}(I_1, I_2, \psi)\, p\right) \\
& +\frac{6}{4!}\frac{\partial^4}{\partial I_2^2 \partial \psi^2}\left(\bar{A}_{2,2,3,3}(I_1, I_2, \psi)\, p\right)
\end{aligned}
\tag{41}
$$

where $p = p(I_1, I_2, \psi)$ and the coefficients for the GFPK equation are

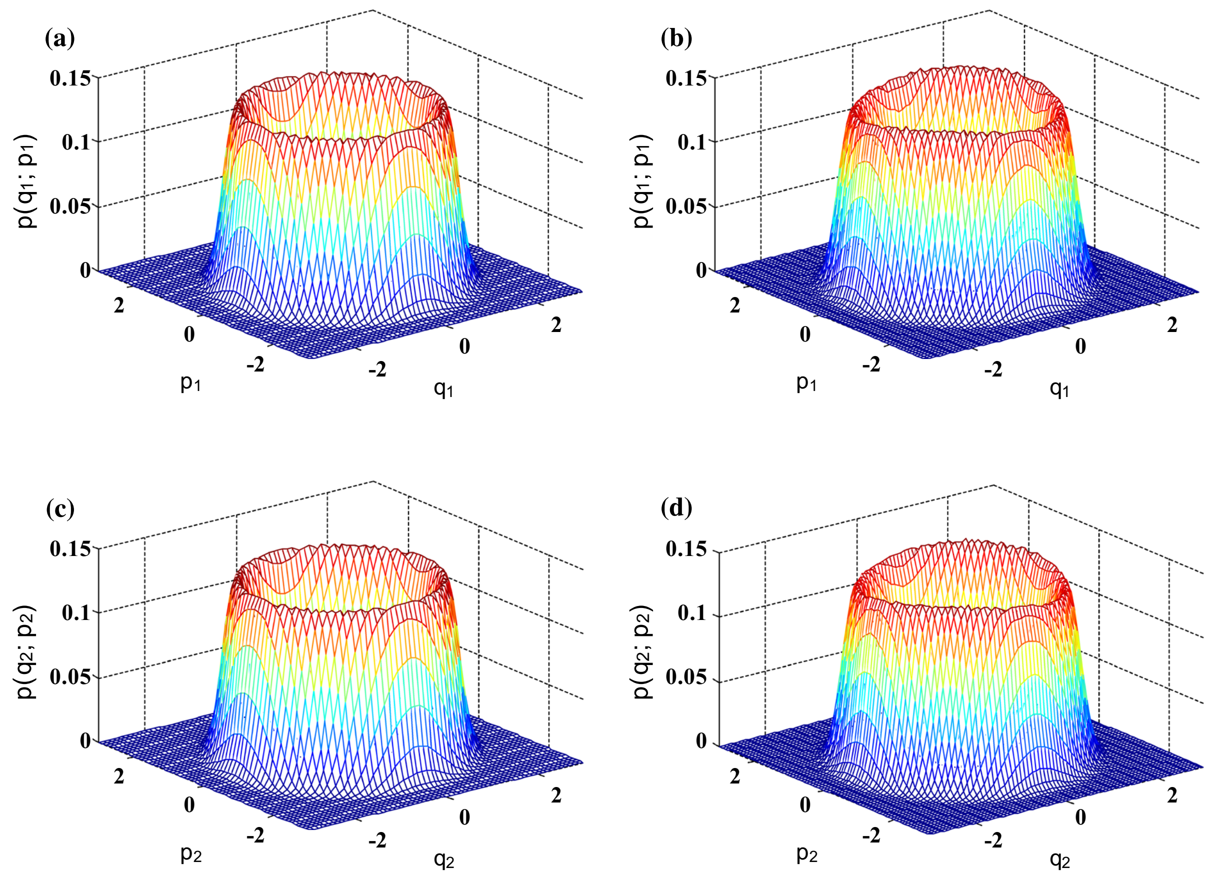

Fig. 7 Stationary joint probability densities $p\,(q_1,\,p_1)$ and $p\,(q_2,\,p_2)$ of system in resonant case. **a** ,**c** analytical results and **b**, **d** results from the Monte Carlo simulation

$$\begin{aligned}
\bar{A}_1 &= -\varepsilon^2\alpha_{11}I_1 - \frac{3}{2}\varepsilon^2\alpha_{12}\omega_1 I_1^2 - \varepsilon^2\beta_1\sqrt{I_1 I_2}\cos\psi \\
&\quad + \frac{\varepsilon^2\left(2D_{11}+\lambda_1 E[Y_1^2]\right)}{2\omega} \\
\bar{A}_2 &= -\varepsilon^2\alpha_{21}I_2 - \frac{3}{2}\varepsilon^2\alpha_{22}\omega_2 I_2^2 - \varepsilon^2\beta_2\sqrt{I_1 I_2}\cos\psi \\
&\quad + \frac{\varepsilon^2\left(2D_{22}+\lambda_2 E[Y_2^2]\right)}{2\omega} \\
\bar{A}_3 &= \varepsilon^2\left(\frac{\beta_1}{2}\sqrt{\frac{I_2}{I_1}} + \frac{\beta_2}{2}\sqrt{\frac{I_1}{I_2}}\right)\sin\psi
\end{aligned} \tag{42}$$

$$\begin{aligned}
\bar{A}_{1,1} &= \frac{\varepsilon^2\left(2D_{11}+\lambda_1 E[Y_1^2]\right)}{\omega}I_1 \\
&\quad + \frac{\varepsilon^4\lambda_1 E[Y_1^4]}{4\omega^2}; \\
\bar{A}_{2,2} &= \frac{\varepsilon^2\left(2D_{22}+\lambda_2 E[Y_2^2]\right)}{\omega}I_2 + \frac{\varepsilon^4\lambda_2 E[Y_2^4]}{4\omega^2};
\end{aligned}$$

$$\begin{aligned}
\bar{A}_{3,3} &= \frac{\varepsilon^2}{4\omega}\left(\frac{2D_{11}+\lambda_1 E[Y_1^2]}{I_1} + \frac{2D_{22}+\lambda_2 E[Y_2^2]}{I_2}\right) \\
&\quad + \frac{\varepsilon^4}{32\omega^2}\left(\frac{\lambda_1 E[Y_1^4]}{I_1^2} + \frac{\lambda_2 E[Y_2^4]}{I_2^2}\right)
\end{aligned} \tag{43}$$

otherwise : $\bar{A}_{r_1,r_2} = 0$

$$\begin{aligned}
&\bar{A}_{1,1,1} = \frac{3\varepsilon^4}{2}\frac{\lambda_1 E[Y_1^4]}{\omega^2}I_1;\ \bar{A}_{2,2,2} = \frac{3\varepsilon^4}{2}\frac{\lambda_2 E[Y_2^4]}{\omega^2}I_2; \\
&\quad \bar{A}_{3,3,3} = 0
\end{aligned} \tag{44}$$

otherwise : $\bar{A}_{r_1,r_2,r_3} = 0,$

$$\begin{aligned}
\bar{A}_{1,1,1,1} &= \frac{3\varepsilon^4}{2\omega^2}I_1^2\lambda_1 E[Y_1^4]; \\
\bar{A}_{2,2,2,2} &= \frac{3\varepsilon^4}{2\omega^2}I_2^2\lambda_2 E[Y_2^4]; \\
\bar{A}_{3,3,3,3} &= \frac{3\varepsilon^4}{32}\frac{\lambda_1 E[Y_1^4]}{\omega^2 I_1^2} + \frac{3}{32}\frac{\lambda_2 E[Y_2^4]}{\omega^2 I_2^2}
\end{aligned}$$

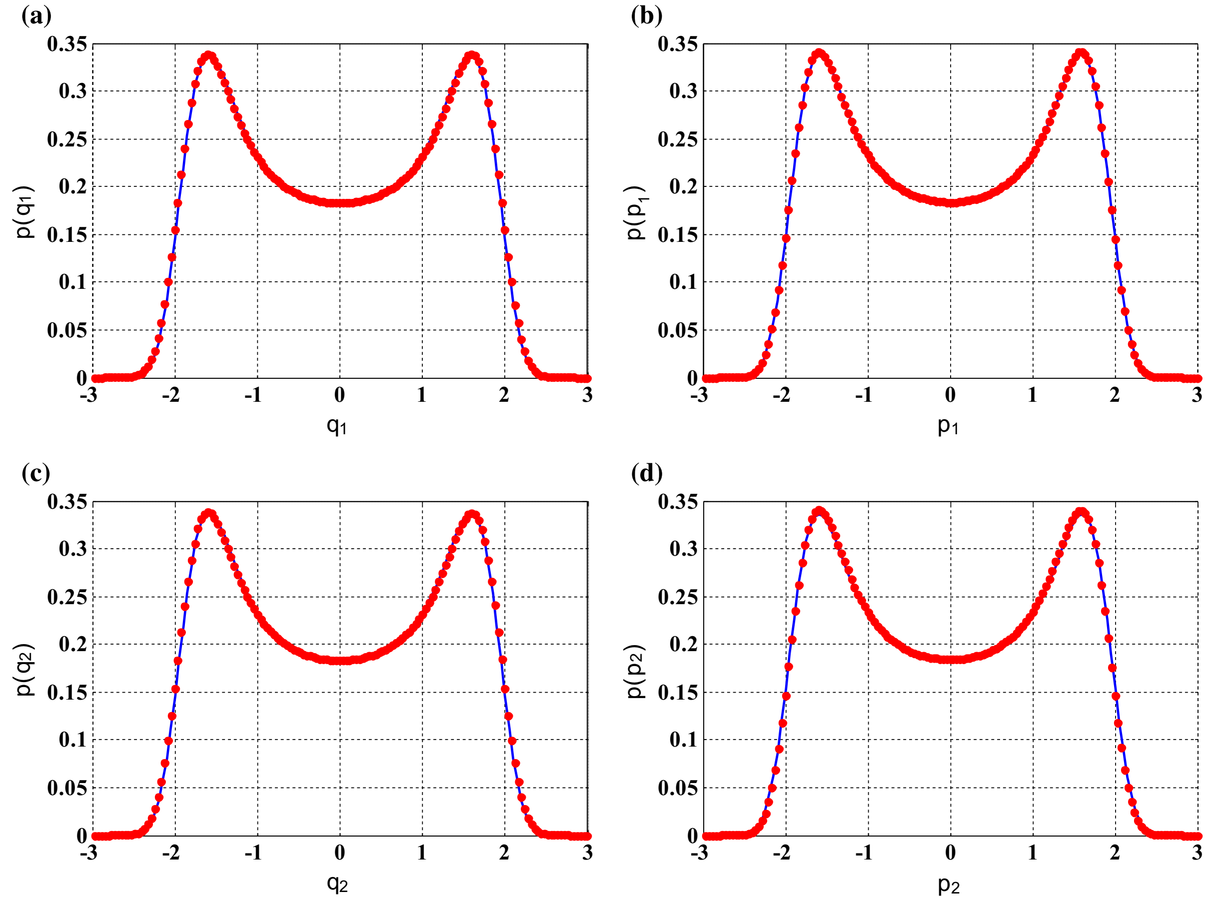

Fig. 8 Stationary marginal probability density functions of q_i and p_i in resonant case. The parameters are the same as those in Fig. 7.

$$\bar{A}_{1,1,3,3} = \frac{\varepsilon^4}{2}\lambda_1 E[Y_1^4];\ \bar{A}_{2,2,3,3} = \frac{\varepsilon^4}{2}\lambda_2 E[Y_2^4] \quad (45)$$

otherwise : $\bar{A}_{r_1,r_2,r_3,r_4} = 0$.

The reduced GFPK equation (41) here cannot be solved theoretically. The finite difference method is used here to solve the reduced averaged GFPK equation [32,33]. In our calculations, the finite difference method is work out with following discrete steps: $\Delta I_1 = 0.1$, $\Delta I_2 = 0.1$, $\Delta\psi = 2\pi/70$.

After obtaining the stationary solution $p(I_1, I_2, \psi)$, the approximate stationary probability density of the displacements and momenta of system (29) is of the form

$$p(q_1, p_1, q_2, p_2) = \frac{1}{2\pi} p(I_1, I_2, \psi). \quad (46)$$

where $\psi = \theta_1 - \theta_2, q_1 = x_1, p_1 = \dot{x}_1, q_2 = x_2, p_2 = \dot{x}_2$.The other statistics of the stationary response of system (29) can then be obtained from $p(I_1, I_2, \psi)$ or $p(q_1, p_1, q_2, p_2)$.

Figures 7 and 8 show some numerical results which are calculated for parameters: $\alpha'_{11} = -6.0, \alpha_{12} = 3.0, \beta_1 = 1.0, \alpha'_{21} = -6.0, \alpha_{22} = 3.0, \beta_2 = 1.0, \eta_{11} = -3.0, \eta_{12} = 3.0, \eta_{21} = -3.0, \eta_{22} = 3.0, \omega'_1 = 1.0, \omega'_2 = 1.0, \varepsilon = 0.1, \tau_1{=}1.0, \tau_2 = 3.0, 2D_{11} = 1.5, 2D_{22} = 1.5, \lambda_1 = 2.5, E[Y_1^2] = 0.2, \lambda_2 = 2.5, E[Y_2^2] = 0.2$. Shown in Fig. 7 are the stationary joint probability density functions of system (29) in resonant case. Figure 7a ,c shows the results obtained by proposed method. Figure 7b, d shows the results from Monte Carlo simulation. Figure 8 shows the stationary marginal density functions of q_i and p_i which can be obtained by integrating $p(q_i, p_i)$. In this figure, the solid lines are the result obtained from the proposed method while the dotted lines are those from Monte Carlo simulation. It is seen from these figures that the

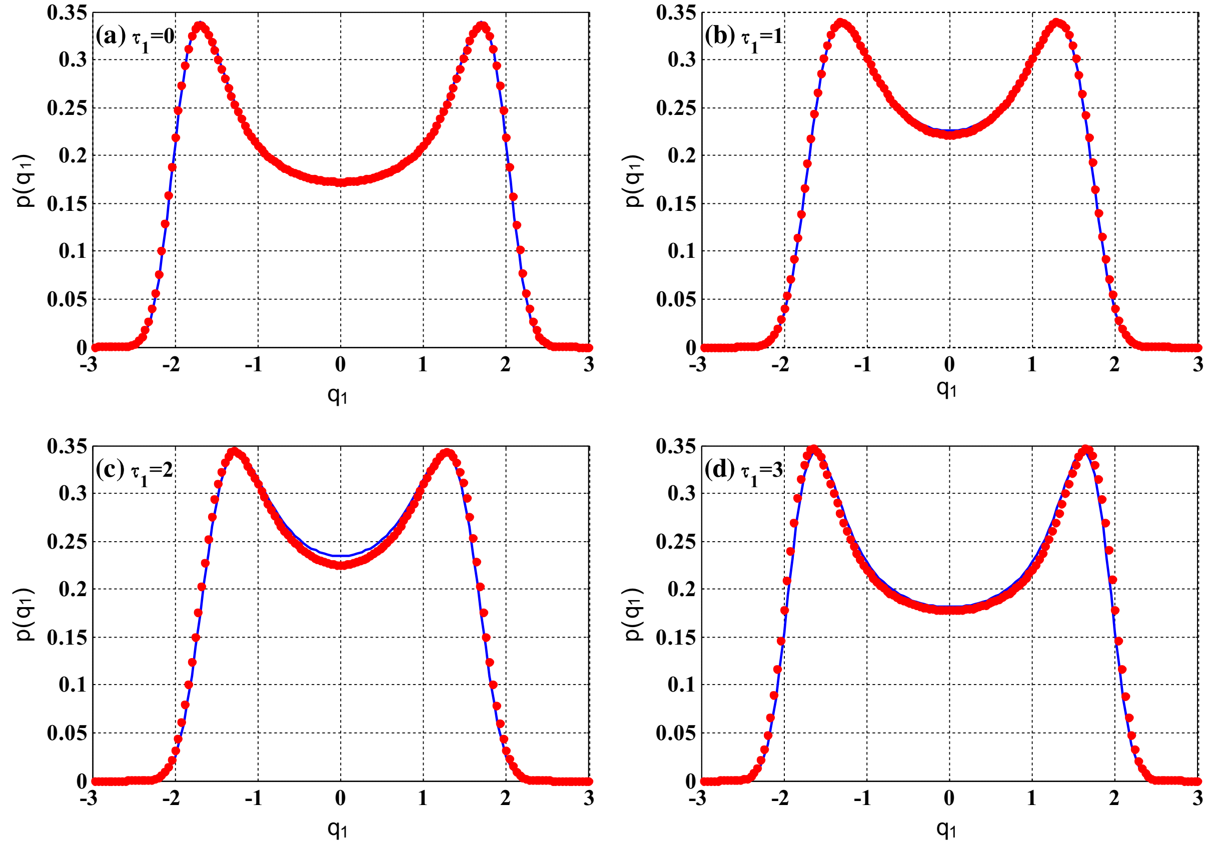

Fig. 9 The influence of τ_1 on the stationary marginal probability density of displacement q_1. **a** $\tau_1 = 0$, $\tau_2 = 2$; **b** $\tau_1 = 1$, $\tau_2 = 2$; **c** $\tau_1 = 2$, $\tau_2 = 2$; **d** $\tau_1 = 3$, $\tau_2 = 2$. The solid lines are the results from proposed method. The dotted line are the results from Monte Carlo simulation. The other parameters are the same as those in Fig. 7.

analytical results agree well with the Monte Carlo simulation. Moreover, in order to show the influence of the time delay parameter τ_1, the stationary probability density functions of system (29) in resonant case are shown in Fig. 9. It can be seen from the figure that the time delay in the feedback control force can affect the bifurcation of the system.

6 Conclusion

In the present paper, we consider a technique for predicting the quasi-integrable Hamiltonian system with multi-time delays under combined Gaussian white noise and Poisson white noise excitation, in which the Gaussian white noise is independent of Poisson white noise excitation. This technique can be viewed as the generalization of the stochastic averaging method. In order to get the response of the systems, two-step approximations are applied. First of all, the time-delayed system state variables are approximated by using the variables without time delay. And the system is transformed to the one with time delay as the parameters. Then, the stochastic averaging method for quasi-integrable Hamiltonian system with combined Gaussian and Poisson white noise can be applied to the transformed system. After the two-step approximation, the averaged SIDE and averaged GFPK equation who governing the probability density of the system response are obtained. The dimension of the original system is reduced from $2n$ to lower dimension ($n$for non-resonant case and $n+\beta$ for resonant case). In order to show the application of the proposed method, two examples are calculated, one for single degree oscillator, and the other for two degree oscillator. After solving the reduced averaged GFPK equations, the stationary probability density functions are derived by using the perturbation technique or finite difference method. The results from Monte Carlo simulation are also calculated to show the validity of the proposed method. In addition, the influences of the time delays on the response of the system are also investigated.

Acknowledgements This study was supported by the National Natural Science Foundation of China under Grants Nos. 11502199, 11372262, 11572247 and the Fundamental Research Funds for Central Universities.

Appendix A

The coefficients of averaged GFPK equation for the non-resonant case:

$$\bar{A}_{r_1}(\mathbf{I}) = \frac{\varepsilon^2}{(2\pi)^n}\int_0^{2\pi}\left(-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial H}{\partial p_j}\frac{\partial I_{r_1}}{\partial p_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n_g}\sigma_{i,k}\sigma_{j,k}\frac{\partial^2 I_{r_1}}{\partial p_i\partial p_j}\right)\mathrm{d}\boldsymbol{\theta} + \sum_{k=2}^{u}\varepsilon^k\sum_{l=1}^{n_p}\frac{\lambda_l E[Y_l^k]}{(2\pi)^n}\int_0^{2\pi}A_{r;k;l}\mathrm{d}\boldsymbol{\theta}, \tag{47}$$

$$\bar{A}_{r_1,r_2}(\mathbf{I}) = \frac{\varepsilon^2}{(2\pi)^n}\int_0^{2\pi}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n_g}\frac{\partial I_{r_1}}{\partial p_i}\frac{\partial I_{r_2}}{\partial p_j}\sigma_{i,k}\sigma_{j,k}\right)\mathrm{d}\boldsymbol{\theta} + \sum_{k=2}^{u}\varepsilon^k\sum_{l=1}^{n_p}\frac{\lambda_l E[Y_l^k]}{(2\pi)^n}\int_0^{2\pi}\left(\sum_{k_1+k_2=k}A_{r_1;k_1;l}A_{r_2;k_2;l}\right)\mathrm{d}\boldsymbol{\theta}, \tag{48}$$

$$\bar{A}_{r_1,r_2,\ldots,r_j}(\mathbf{I}) = \sum_{k=j}^{u}\varepsilon^k\sum_{l=1}^{n_p}\frac{\lambda_l E[Y_l^k]}{(2\pi)^n}\int_0^{2\pi}\left(\sum_{k_1+k_2+\cdots+k_j=k}A_{r_1;k_1;l}A_{r_2;k_2;l}\cdots A_{r_j;k_j;l}\right)\mathrm{d}\boldsymbol{\theta}, \tag{49}$$

where $A_{r;k;l} = A_{r;k;l}(\mathbf{q},\mathbf{p})$ given in Ref. [23], $\mathbf{I} = [I_1, I_2, \ldots, I_n]^T$, $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_n]^T$ and $\int_0^{2\pi}[\cdot]\,\mathrm{d}\boldsymbol{\theta} = \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}[\cdot]\,\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_n$ denotes the n-fold integral.

Appendix B

The coefficients of averaged GFPK equation for resonant case :

$$\bar{A}_{r_1}(\mathbf{I},\boldsymbol{\psi}) = \frac{\varepsilon^2}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(-\sum_{i,j=1}^{n}m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial I_{r_1}}{\partial p_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{n_g}\sigma_{ik}\sigma_{jk}\frac{\partial^2 I_{r_1}}{\partial p_i\partial p_j}\right)\mathrm{d}\boldsymbol{\theta}_1 + \sum_{k=1}^{u}\sum_{l=1}^{n_p}\frac{\varepsilon^k\lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}}\int_0^{2\pi}A_{r_1;k;l}\mathrm{d}\boldsymbol{\theta}_1 \tag{50}$$

$$\bar{A}_{n+v_1}(\mathbf{I},\boldsymbol{\psi}) = \frac{1}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left[\mathrm{O}\left(\varepsilon^2\right) + \varepsilon^2\left(-\sum_{i,j=1}^{n}m_{ij}\frac{\partial H}{\partial p_j}\frac{\partial \psi_{v_1}}{\partial p_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{n_g}\sigma_{ik}\sigma_{jk}\frac{\partial^2\psi_{v_1}}{\partial p_i\partial p_j}\right)\right]\mathrm{d}\boldsymbol{\theta}_1 + \sum_{k=1}^{u}\sum_{l=1}^{n_p}\frac{\varepsilon^k\lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}}\int_0^{2\pi}C_{v_1;k;l}\mathrm{d}\boldsymbol{\theta}_1 \tag{51}$$

$$\bar{A}_{r_1,r_2}(\mathbf{I},\boldsymbol{\psi}) = \frac{\varepsilon^2}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(\sum_{i,j=1}^{n}\sum_{k=1}^{n_g}\frac{\partial I_{r_1}}{\partial p_i}\frac{\partial I_{r_2}}{\partial p_j}\sigma_{i,k}\sigma_{j,k}\right)d\boldsymbol{\theta}_1 + \sum_{k=2}^{u}\sum_{l=1}^{n_p}\frac{\varepsilon^k\lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(\sum_{k_1+k_2=k}A_{r_1;k_1;l}A_{r_2;k_2;l}\right)\mathrm{d}\boldsymbol{\theta}_1 \tag{52}$$

$$\bar{A}_{r_1,n+v_1}(\mathbf{I},\boldsymbol{\psi}) = \frac{\varepsilon^2}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(\sum_{i,j=1}^{n}\sum_{k=1}^{n_g}\frac{\partial I_{r_1}}{\partial p_i}\frac{\partial \psi_{v_1}}{\partial p_j}\sigma_{i,k}\sigma_{j,k}\right)\mathrm{d}\boldsymbol{\theta}_1 + \sum_{k=2}^{u}\sum_{l=1}^{n_p}\frac{\varepsilon^k\lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(\sum_{k_1+k_2=k}A_{r_1;k_1;l}C_{v_1;k_2;l}\right)\mathrm{d}\boldsymbol{\theta}_1 \tag{53}$$

$$\bar{A}_{n+v_1,n+v_2}(\mathbf{I},\boldsymbol{\psi}) = \frac{\varepsilon^2}{(2\pi)^{n-\alpha}}\int_0^{2\pi}\left(\sum_{i,j=1}^{n}\sum_{k=1}^{n_g}\sigma_{i,k}\sigma_{j,k}\frac{\partial \psi_{v_1}}{\partial p_i}\frac{\partial \psi_{v_2}}{\partial p_j}\right)\mathrm{d}\boldsymbol{\theta}_1 + \sum_{k=2}^{u}\sum_{l=1}^{n_p}\frac{\varepsilon^k\lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}}\int_0^{2\pi}$$

$$\left(\sum_{k_1+k_2=k} C_{v_1;k_1;l} C_{v_2;k_2;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{54}$$

$$\bar{A}_{r_1,r_2,r_3}(\mathbf{I},\boldsymbol{\psi}) = \sum_{k=3}^{u}\sum_{l=1}^{n_p} \frac{\varepsilon^k \lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(\sum_{k_1+k_2+k_3=k} A_{r_1;k_1;l} A_{r_2;k_2;l} A_{r_3;k_3;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{55}$$

$$\bar{A}_{r_1,r_2,n+v_1}(\mathbf{I},\boldsymbol{\psi}) = \sum_{k=3}^{u}\sum_{l=1}^{n_p} \frac{\varepsilon^k \lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(\sum_{k_1+k_2+k_3=k} A_{r_1;k_1;l} A_{r_2;k_2;l} C_{v_1;k_3;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{56}$$

$$\bar{A}_{r_1,n+v_1,n+v_2}(\mathbf{I},\boldsymbol{\psi}) = \sum_{k=3}^{u}\sum_{l=1}^{n_p} \frac{\varepsilon^k \lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(\sum_{k_1+k_2+k_3=k} A_{r_1;k_1;l} C_{v_1;k_2;l} C_{v_2;k_3;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{57}$$

$$\bar{A}_{n+v_1,n+v_2,n+v_3}(\mathbf{I},\boldsymbol{\psi}) = \sum_{k=3}^{u}\sum_{l=1}^{n_p} \frac{\varepsilon^k \lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(\sum_{k_1+k_2+k_3=k} C_{v_1;k_1;l} C_{v_2;k_2;l} C_{v_3;k_3;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{58}$$

$$\bar{A}_{r_1,\ldots r_{j-s},n+v_1,\ldots,n+v_s}(\mathbf{I},\boldsymbol{\psi}) = \sum_{k=j}^{u}\sum_{l=1}^{n_p} \frac{\varepsilon^k \lambda_l E[Y_l^k]}{(2\pi)^{n-\alpha}} \int_0^{2\pi} \left(\sum_{k_1+k_2+\cdots+k_j=k} A_{r_1;k_1;l} \cdots A_{r_s;k_s;l} C_{v_1;k_{s+1};l} \cdots C_{v_{j-s};k_j;l}\right) \mathrm{d}\boldsymbol{\theta}_1 \tag{59}$$

$s = 0, \ldots, j,\ j = 4, \ldots, u,\ r_i = 1, \ldots, n;\ v_i = 1, \ldots, \alpha.$

where $\mathbf{I} = [I_1, \ldots, I_n]^T$, $\psi = [\psi_1, \ldots, \psi_\alpha]^T$ and $\int_0^{2\pi} [\cdot]\mathrm{d}\boldsymbol{\theta}_1 = \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} [\cdot]\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_\alpha$ is the $n-\alpha$-fold integral notation. The terms $A_{r;k;l} = A_{r;k;l}(\mathbf{q},\mathbf{p})$ and $C_{r;k;l} = C_{r;k;l}(\mathbf{q},\mathbf{p})$ are given in Ref. [24].

Appendix C

Substituting this solution (27) to Eq. (25) and collecting the terms of same order of ε, the equations that p_0, p_1 and p_2 satisfy are

$$\varepsilon^2: \quad 0 = -\frac{\partial}{\partial I}\left(\bar{A}_1(I)\,p_0\right) + \frac{1}{2}\frac{\partial}{\partial I^2}\left(\bar{A}_2^{(1)}(I)\,p_0\right) \tag{60}$$

$$\varepsilon^3: \quad 0 = -\frac{\partial}{\partial I}\left(\varepsilon\bar{A}_1(I)\,p_1\right) + \frac{1}{2}\frac{\partial^2}{\partial I^2}\left(\varepsilon\bar{A}_2^{(1)}(I)\,p_1\right) \tag{61}$$

$$\varepsilon^4: \quad 0 = -\frac{\partial}{\partial I}\left(\varepsilon^2\bar{A}_1(I)\,p_2\right) + \frac{1}{2}\frac{\partial^2}{\partial I^2}\left(\varepsilon^2\bar{A}_2^{(2)}(I)\,p_2\right) - \frac{1}{3!}\frac{\partial^3}{\partial I^3}\left(\bar{A}_3(I)\,p_0\right) + \frac{1}{4!}\frac{\partial^4}{\partial I^4}\left(\bar{A}_4(I)\,p_0\right) \tag{62}$$

$$\vdots$$

where

$$\bar{A}_2^{(1)}(I) = \varepsilon^2\frac{2D + \lambda E\left[Y^2\right]}{\omega} I;$$

$$\bar{A}_2^{(2)}(I) = \varepsilon^4\frac{\lambda E\left[Y^4\right]}{4\omega^2} \text{ and}$$

$$\bar{A}_2(I) = \bar{A}_2^{(1)}(I) + \bar{A}_2^{(2)}(I)$$

One can get $p_0, p_1, p_2, \ldots$ by solving Eqs. (60)–(62) step by step. And, the solution (27) can be obtained.

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